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Making Things Simpler in Circuit Theory*

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An innovative way of teaching introductory circuit theory to higher education first courses of non-electrical engineering students arises from our own teaching experience. In these students' learning programme, circuit theory is located at the very early stages (first or second semester, when the student is still not too skilled on maths). Fewer credits are being devoted to it, since it is viewed as an introductory subject, a precursor to subsequent systems theory and electronics. In this paper we wish to encourage teachers to give up the classical approach to circuits and replace it with the proposed Laplace transform approach which, remembering A. Einstein's sentence, 'Everything should be made as simple as possible' allows circuit theory to be made much simpler. Perhaps, this attitude could grow into a text book, definitively new and different, intended for teaching the basics of circuits to first courses of non-electrical engineering students.

Keywords: teaching and learning basic circuit theory; differential equations for time responses; phasors for steady-state; Laplace transform for frequency response

INTRODUCTION

IT TAKES NO TIME to enter the amazon.com web page and search for books under keywords 'Electric Circuits/Networks' and 'Analysis/Theory' to end up, even leaving aside those devoted exclusively to problem solving and computer simulation, with almost a hundred different titles, many of them in successive new editions (up to the tenth). References at the end of this paper are far from including all of them: they are just a small sample, but representative enough because, after checking the contents of about fifty of them, none has been found with the simplifying methodology proposed in this paper.

Typically these text books are intended for either the junior or sophomore year of undergraduate electrical or computer engineering, so that most of them share the same contents, perspective and methods. Making an attempt at organizing their content, six main conceptual blocks could be distinguished, as in Table 1.

Where the more classical text books [1, 2] have mostly changed through the years is in their tendency to include state-of-the-art devices (operational amplifiers), tools (computer simulators) and trends (services on line and e-learning support). But contents, structure, mathematical tools and teaching methodology remain almost the same. As clearly exhibited by reference [3] title, once the inductor and the capacitor and their differential current-voltage characteristics have been introduced, three different mathematical techniques are presented:

1. Transient responses in first- and second-order circuits (block iii) are studied turning to time domain differential equations;
2. Sinusoidal steady-state analysis (block iv) is performed through the concept of phasor and therefore involving complex exponential functions;
3. Frequency domain analysis and responses (block v) are developed with the Laplace transform tool.

When not completely avoided [4–6], the Laplace transform is left to the end of the book [3, 7–14] and, therefore, to the end of the course.

Although such a schema is assumed to be well suited for electrical engineers, there are some other technical engineering disciplines where this framework hardly fits, either due to the reduced number of credits allocated for the course or to the special bias in the training. For example, at the Public University of Navarre, five different Engineering Degrees (Higher Degree in Telecommunication Engineering, Diploma in Sound and Image Technical Engineering, Diplomas in Mechanical Engineering and Electrical Engineering and Higher Degree in Industrial Engineering) are offered, including basic circuit theory as a compulsory subject in the very early educational stages [15]. Even though each one is specifically focused on its own topics and training purposes, the lack of time to cover all five first blocks is a typical problem.

Our circuit teaching experience in the two first engineering degrees indicated in the previous paragraph, following the classical structure of the text books, reveals some unpleasant drawbacks:

- First, analysis techniques and theorems introduced in block ii) for the time-domain have to be repeated later for phasorial notation and then again for complex-frequency notation, which is time consuming and rather boring.
- Second, as Laplace transform (LT) is at the end, it becomes the most probable subject to be

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Table 1. Electric circuits, what the literature contains

	Basic block	Contents	Mathematical tools
i	Linear analysis	Introduction. Electrical magnitudes. Basic concepts. Kirchoff's laws. Independent and dependent sources. Resistances. Series-parallel R circuits. Voltage and current division. Operational amplifier.	Time functions integration and differentiation. Notions of trigonometry.
ii	Analysis techniques	Node and loop analysis. Network theorems.	Equation systems solving. Matrix arithmetic.
iii	Time domain analysis	Inductors, capacitors and duality. First order RL and RC circuits. Second order linear circuits.	Methods to solve first and second order differential equations.
iv	Sinusoidal steady-state analysis	Sinusoidal steady-state analysis by phasors and power calculation. Frequency response.	Calculus with complex numbers and exponentials. Notions of trigonometry.
v	Frequency domain analysis	Laplace transform analysis: basics and applications. Transfer functions. Principles of first and second order filters. Poles and zeroes. Stability. Frequency response and Bode diagrams.	Laplace transform: direct and inverse. Complex functions polar expression: magnitude and phase.
vi	Additional chapters	Transformers and magnetically coupled circuits. Two ports. Interconnection of two ports. Polyphase circuits. Fourier series and transform. Resonant and bandpass circuits.	

discarded whenever the course runs short of time, which is a pity given its connection to Fourier transform (for spectrum analysis and modulation processes) and z-transform (for digital signal processing).

- Third, even though sinusoidal signals are essential in many fields, from electric power generation and transmission to satellite communication and control, telecommunication engineers must face and deal with many other different types of signals (steps, pulses, impulses . . .) that do not allow the phasor steady-state treatment.
- Fourth, this methodology leads to an apparent disconnection between time response, related to natural frequencies and damped behaviours explained in block iii), and frequency response, connected to transfer functions and Bode diagrams introduced in blocks iv) and v) when, in the end, both of them are a direct consequence of the system poles location.

There are other drawbacks too, which highlight the benefits arising from our alternative proposal based on the use of the Laplace transform as the unique mathematical tool needed to analyse circuit performance in domains of time and frequency, with no loss of information and taking much less mathematical effort and saving time. The success of this approach is borne out by three years' experience of teaching circuits in this way, after ten years of facing the problems arising from traditional methods.

THE LAPLACE TRANSFORM TOOL

Outlining some of the advantages:

- Analysis related to blocks iii), iv) and v) can be more comfortably and easily performed just from the s-domain transformed circuit.
- Assuming that students know how to perform direct and inverse transforms, extraction of time response and transient analysis is developed without having to solve differential equations, (DE).
- Using an innovative procedure (exploited only rarely [3, 7, 8] and never in its entirety), the underlying connection between complex-frequency network functions and sinusoidal steady-state performance is presented and the way to avoid the use of phasors and complex exponentials is explained.
- From here, steady-state output and frequency response are easily derived from the transformed circuit analysis.

More specifically, take a linear circuit containing one, two or more energy-storage components. At a given time $t = t_o$ a sudden change occurs in the circuit, so that the time-invariance condition is not fulfilled and therefore two different network analyses (before and after the change) are required.

The following steps describe the systematic methodology we propose to be applied when solving circuit problems under these conditions in order to obtain as much information as possible about the output response in the time domain $y(t > t_o)$, and about the circuit performance in the frequency domain, but minimizing the amount of mathematical knowledge required.

STEP LT.1—Initial conditions calculation. The circuit before the change has to be analysed in order to obtain the value at $t = t_o^-$ of every circuit variable physically subjected to continuity, i.e. voltages across capacitors and currents through inductors. These are the only magnitudes whose values remain compulsorily unchanged in a non-time-invariance situation.

STEP LT.2—Circuit transformation into the *s*-domain. The circuit topology after the change (for $t > t_o$) has to be transformed into the complex-frequency domain. When replacing the energy-storage elements by their respective impedances $Z_k(s)$, the previously calculated initial conditions are to be incorporated through the adequate internal sources, either voltage sources in series or current sources in parallel, that represent the energy provided to the network by the reactive components due to their memory capability.

STEP LT.3—Analysis in the transformed *s*-domain. Whichever way the unknown output magnitude $y(t)$ is defined and for any analysis technique used, the presence of external sources (those that supply network excitation) and internal sources (those that, according to step LT.2, reflect initial conditions) allows an easy recognition between the zero-input $Y_{zi}(s)$ and the zero-state $Y_{zs}(s)$ contributions to the transformed output variable $Y(s)$, even if superposition principle is not applied. This analysis will always yield an expression for the output transform $Y(s)$ as a ratio between two polynomials in the s complex-frequency variable, that can be written as:

$$Y(s) = \frac{n(s)}{d(s)} = Y_{zs}(s) + Y_{zi}(s) = \underbrace{\frac{N(s)}{D(s)} \cdot X(s)}_{\text{zero-state}} + \underbrace{\frac{1}{D(s)} \sum_k IC_k(s)}_{\text{zero-input}} \quad (1)$$

Here $D(s)$ is obviously the system characteristic polynomial, $IC_k(s)$ carries the information related to the initial condition at the k–th reactive element, and the sum in k is extended over all the energy-storage elements contained in the circuit. $X(s)$ represents the Laplace transform of the single external source $x(t)$.[1]

STEP LT.4— Inverse Laplace transform computation. Instead of a mathematical distinction between natural and forced responses (both depending on the excitation source), this method provides the complete response naturally split into zero-input and zero-state components, with the advantage of knowing the excitation source is not needed to compute the former, but just for the latter.

Such distinction between zero-input and zero-state responses in the time domain is obtained when the inverse transform of both contributions in (1) is calculated separately,[2] through the corresponding partial-fraction expansion,

$$y(t > t_o) = y(t > t_o)|_{zs} + y(t > t_o)|_{zi} = LT^{-1}[Y_{zs}(s)] + LT^{-1}[Y_{zi}(s)] \quad (2)$$

STEP LT.5—Sinusoidal steady-state response. As a special case, it is next considered that the steady-state is reached with the circuit excited by sinusoidal sources which, when transforming into the complex-frequency domain, are replaced by their Laplace transform (instead of by the corresponding phasor), making use of the well-known transformed pairs:

$$\left.\begin{aligned} x(t) = K\cos\omega t\, u(t) \rightarrow X(s) = LT[x(t)] = \frac{Ks}{s^2+\omega^2} \\ x(t) = K\sin\omega t\; u(t) \rightarrow X(s) = LT[x(t)] = \frac{K\omega}{s^2+\omega^2} \end{aligned}\right\} \quad (3)$$

Whenever the circuit under analysis contains several sinusoidal sources of the same frequency but not in phase, which is a quite ordinary situation, the best and shortest way to proceed to achieve their Laplace transform (it is not usual to find this pair in tables) is to turn to trigonometric relations and the above transformed pairs in (3) to get the most general expression,

$$\begin{aligned} x(t) &= K\cos(\omega t + \phi)\, u(t) \\ &= K[\cos\omega t \cos\phi - \sin\omega t \sin\phi]\; u(t) \\ &\Rightarrow \quad X(s) = LT[x(t)] \\ &= \frac{Ks\cos\phi - K\omega\sin\phi}{s^2+\omega^2} \\ &= \frac{M(s)}{(s-j\omega)(s+j\omega)} \end{aligned} \quad (4)$$

that should be used as the ac steady-state input transform (understanding that angle ϕ can be either positive or negative) when computing the zero-state contribution in (1). Only the zero-state contribution is of interest since the final aim here is to obtain the forced response; a zero-input component only renders a natural response so that it can be neglected in (1),

$$\begin{aligned} Y(s) &= Y_{zs}(s) = \frac{N(s)}{D(s)} \cdot X(s) = H(s) \cdot X(s) \\ &= H(s) \cdot \frac{M(s)}{(s-j\omega)(s+j\omega)} \end{aligned} \quad (5)$$

On the one hand, energy-storage elements could be replaced just by their respective impedances if only ac steady-state is pursued, which means that step LT.1 can be completely avoided and step LT.2 considerably simplified.

When (5) is expanded into partial-fractions, just one of them yields the forced response $Y_{ss}(s)$, and the remaining ones produce the natural response,

$$Y(s) = Y_{zs}(s) = \underbrace{\frac{As+B}{(s^2+\omega^2)}}_{\text{forced}} + \underbrace{\sum_i \sum_{m=1}^{M_i} \frac{D_{i,m}}{(s-p_i)^m} + \sum_k \sum_{m=1}^{M_k} \frac{E_{k,m}s + F_{k,m}}{\left((s-\alpha_k)^2 + \beta_k^2\right)^m}}_{\text{natural}} \quad (6)$$

Here, in the transformed natural component expression, the sum extended over i takes account of the system p_i real poles, with possible multiplicity M_i, meanwhile the sum over k covers those poles forming complex-conjugate pairs ($\alpha_k \pm j\beta_k$), with possible multiplicity M_k. Since this is the natural component of the zero-state contribution, it is related to the inertial behaviour of reactive elements and is therefore called the natural response inertial component.

To neglect this natural component means there is no need to calculate its constants, but just A and B, because the steady-state contribution will be always a recognizable partial-fraction in (6) carrying information about excitation frequency (it is assumed the circuit is not an oscillator). The easiest way to derive A and B values is applying the residues method,

$$Y_{ss}(s) = \frac{As+B}{s^2+\omega^2} = \frac{(R_1+R_2)s + j\omega(R_1-R_2)}{s^2+\omega^2} = \frac{R_1}{s-j\omega} + \frac{R_2}{s+j\omega} \quad (7)$$

Where, according to (5), the residues are computed as,

$$\left.\begin{aligned} R_1 &= (s-j\omega)Y_{zs}(s)|_{s=j\omega} = (s-j\omega)H(s)X(s)|_{s=j\omega} = \frac{H(j\omega)M(j\omega)}{j2\omega} \\ R_2 &= (s+j\omega)Y_{zs}(s)|_{s=-j\omega} = (s+j\omega)H(s)X(s)|_{s=-j\omega} = \frac{H(-j\omega)M(-j\omega)}{-j2\omega} \end{aligned}\right\} \quad (8)$$

Since $H(\pm j\omega)$ and $M(\pm j\omega)$ are complex functions, it is convenient to express them in polar form. At the view of (4),

$$\frac{M(j\omega)}{j2\omega} = \frac{\omega K(j\cos\phi - \sin\phi)}{j2\omega} = \frac{K}{2}\cos\phi + j\frac{K}{2}\sin\phi = \frac{K}{2}\cdot e^{j\phi} = \frac{K}{2}\angle\phi$$

$$\frac{M(-j\omega)}{-j2\omega} = \frac{\omega K(-j\cos\phi - \sin\phi)}{-j2\omega} = \frac{K}{2}\cos\phi - j\frac{K}{2}\sin\phi = \frac{K}{2}\cdot e^{-j\phi} = \frac{K}{2}\angle-\phi$$

Circuit transfer function $H(s)$ exhibits conjugate symmetry,[3] so that it is verified $H(-j\omega) = H^*(j\omega)$ and therefore,

$$H(j\omega) = |H(j\omega)|\cdot e^{j\angle H(j\omega)} = |H(j\omega)|\angle H(j\omega)$$
$$H(-j\omega) = |H(-j\omega)|\cdot e^{j\angle H(-j\omega)} = H^*(j\omega) = |H(j\omega)|\angle - H(j\omega)$$

Carrying these polar expressions into residues equations (8) and then to coefficients A and B in (7), leads to

$$A = R_1 + R_2 = H(j\omega)\frac{K}{2}e^{j\phi} + H(-j\omega)\frac{K}{2}e^{-j\phi} = K|H(j\omega)|\ \cos[\phi + \angle H(j\omega)]$$

$$B = j\omega(R_1 - R_2) = j\omega\left[H(j\omega)\frac{K}{2}e^{j\phi} - H(-j\omega)\frac{K}{2}e^{-j\phi}\right] = -\omega K|H(j\omega)|\ \sin[\phi + \angle H(j\omega)]$$

From here the output steady-state (t greater than zero) response is finally obtained as the inverse LT transform of (7),

$$\begin{aligned} y_{ss}(t) &= LT^{-1}[Y_{ss}(s)] = LT^{-1}\left[\frac{As+B}{s^2+\omega^2}\right] \\ &= A\cos\omega t + \frac{B}{\omega}\sin\omega t = \\ &= K|H(j\omega)|, [\cos\omega t\cos[\phi + \angle H(j\omega)] \\ &\quad - \sin\omega t\sin[\phi + \angle H(j\omega)]] = \\ &= K|H(j\omega)|\ \cos[\omega t + \phi + \angle H(j\omega)] \end{aligned} \quad (9)$$

In the end, the steady-state circuit output response is a sinusoidal waveform with the form $y_{ss}(t) = K'\cos(\omega t + \phi')$ where amplitude K' and phase ϕ' are easily obtained from the input waveform when the circuit transfer function $H(s)$ is evaluated at $s = j\omega$, as shown in the diagram of Fig. 1, where ω is the value of the excitation frequency.

Application of such a procedure to circuits containing a single excitation source is extremely easy, as will be shown later. Nevertheless, for circuits containing several sinusoidal sources (assuming different amplitudes and phases but equal frequencies), some caution is required.[4] In such circuits the steady-state transformed function in (7) might be replaced by,

$$Y_{ss}(s) = \sum_m \frac{A_m s + B_m}{s^2+\omega^2} = \sum_m \left(\frac{R_{1m}}{s-j\omega} + \frac{R_{2m}}{s+j\omega}\right)$$

and every single term in the sum is inversely transformed to render a sinusoidal waveform once the corresponding coefficients A_m and B_m are determined.

STEP LT.6—Frequency response function. Following the above procedure, it is evident that the frequency response function $H(j\omega)$ is obtained from the transfer function $H(s)$ under replacement $s = j\omega$, where ω is a real variable. From here, extracting magnitude response $|H(j\omega)|$ and phase response $\angle H(j\omega)$ and sketching the corresponding Bode plots is a straightforward task.

No further interest will be shown in this last step, since the concepts of transfer function and frequency response are always connected to Laplace transform in all classical circuit text books. It should be remembered that our final aim is to show the way to use this tool, the Laplace transform, to extract complete responses in the time-domain (instead of solving differential equations) and sinusoidal steady-state performance (avoiding complex exponentials and phasors).

THE USUAL APPROACH TO TIME ANALYSIS. DRAWBACKS

The systematic procedure to analyse in the time domain, by writing and solving differential equations, the performance of circuits that contain

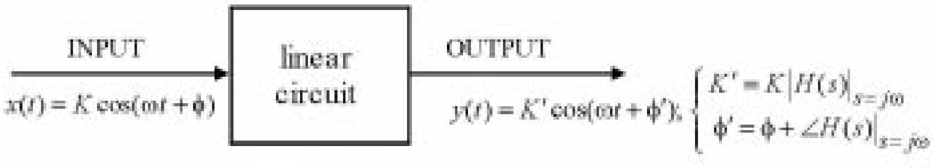


Fig. 1. Conceptual scheme of the LT method applied to ac steady-state.

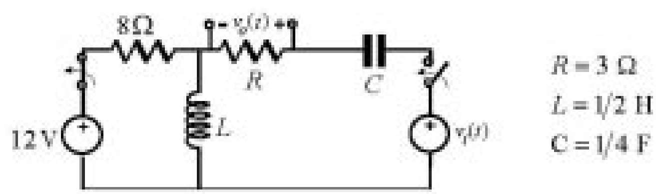


Fig. 2. Second-order circuit chosen as example.

energy-storage components can be found in any of the text books in references, and it is assumed they will be well known to the reader. At the same time it is assumed that solving differential equations requires some extra mathematical skills and knowledge and that this method intrinsically renders the distinction between natural and forced responses (homogeneous and particular solutions for the differential equation).

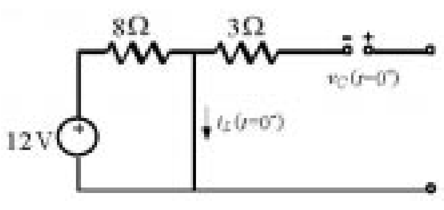
In what follows, such a procedure is applied, step by step, to analyse the simple second-order circuit in Fig. 2. Second-order has been selected for the example because of being fairly representative: identical methodology could be applied to solve any reactive circuit of higher-order. Method suitability regarding to higher-order will be discussed later, when referring to this technique disadvantages.

In the example circuit the output variable $v_o(t)$ is defined as a voltage drop across a resistance and, therefore, it is not a magnitude physically subjected to continuity. A sudden change is introduced at time $t = t_o$, where $t_o = 0$ s is chosen for the sake of simplicity, since the switch on the left is opened at the same time the other one closes. The final aim is to proceed step by step in order to obtain an expression for the output response in the time domain, so that the main drawbacks of this mathematical method can be pointed out and a comparison with our alternative proposal carried out.

STEP DE.1—Initial conditions. As the circuit before the change is excited by a DC voltage source, assuming the steady-state is reached, both reactive components are fully charged, i.e. the capacitor behaves as an open-circuit (no current through it) meanwhile the inductor performs as a short-circuit (no voltage drop across it). The equivalent circuit topology, shown in Fig. 3, exhibits an easy analysis to extract both initial conditions.

STEP DE.2—Differential equation. For any output magnitude, it is always preferable and easier to derive the differential equation for $t > t_o$ in a variable subjected to continuity. In this case the selected variable is the capacitor voltage $v_C(t)$, since the circuit topology is clearly suitable for a mesh analysis. Taking into account that all the components are connected in series and the equations in Fig. 4 relating voltage and current, application of Kirchhoff's Voltage Law is immediate,

$$\begin{aligned}\mathrm{KVL}(t>0): v_i(t) &= v_C(t) + v_R(t) + v_L(t)\\ &= v_C(t) + R\, i_C(t) + v_L(t) =\\ &= v_C(t) + RC\frac{dv_C(t)}{dt} + L\frac{di_L(t)}{dt}\\ &= v_C(t) + RC\frac{dv_C(t)}{dt} + LC\frac{d^2v_C(t)}{dt^2}\end{aligned}$$



Initial conditions:

$$i_L(t=0^-) = \frac{12\ \mathrm{V}}{8\,\Omega} = \frac{3}{2}\ \mathrm{A} \quad (10)$$

$$v_C(t=0^-) = 0\ \mathrm{V} \quad (11)$$

Fig. 3. Simplified equivalent circuit for initial conditions calculation.

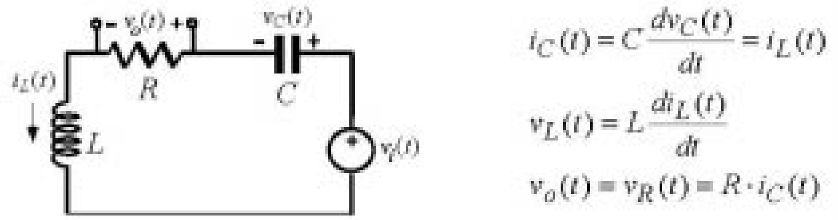


Fig. 4. Circuit in the time domain when $t > 0$ and its current-voltage relations.

Hence, the second-order non-homogeneous differential equation that describes the circuit behaviour in the time domain when $t > 0$ is,

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{1}{LC}v_i(t) \quad (12)$$

Introducing the values for the passive components (given in Fig. 2),

$$\frac{d^2v_C(t)}{dt^2} + 6\frac{dv_C(t)}{dt} + 8v_C(t) = 8v_i(t)$$

STEP DE.3—Natural component. For the homogeneous solution to be achieved, the characteristic equation is derived from the homogeneous differential equation, so that the natural frequencies can be extracted,

$$s^2 + 6s + 8 = 0 \;\Rightarrow\; s_{1,2} = \frac{-6 \pm \sqrt{36-32}}{2}$$
$$= \frac{-6 \pm 2}{2} \;\Rightarrow\; \begin{cases} s_1 = -4 \\ s_2 = -2 \end{cases} \quad (13)$$

Due to the minus sign in both natural frequencies the system stability is ensured and the homogeneous solution (that corresponds to an over-damped behaviour) can be expressed as,

$$\text{Natural component} = v_C(t)|_{natural} = K_1\,e^{s_1 t} + K_2\,e^{s_2 t}$$
$$= K_1\,e^{-4t} + K_2\,e^{-2t} \quad (14)$$

Determination of constants K_1 and K_2 to match initial conditions must be put off to step DE.5

STEP DE.4—Forced component. In order to obtain the particular solution, at this point it is compulsory to know the excitation mathematical function. Assuming, for example, $v_i(t) = e^{-3t}$ the second-order differential equation becomes

$$\frac{d^2v_C(t)}{dt^2} + 6\frac{dv_C(t)}{dt} + 8v_C(t) = 8v_i(t) = 8\,e^{-3t}$$

where the particular solution for trial is

$$v_C(t)|_{forced} = Ke^{-3t} \;\Rightarrow\; \frac{dv_C(t)}{dt} = -3Ke^{-3t}$$

and

$$\frac{d^2v_C(t)}{dt^2} = 9Ke^{-3t}$$

which results into $9K - 18K + 8K = -K = 8 \Rightarrow K = -8$

$$\text{Forced component} = v_C(t)|_{forced} = -8e^{-3t} \quad (15)$$

STEP DE.5—Complete solution. Adding natural component in (14) and forced component in (15) the complete solution for $v_C(t \geq 0)$ is obtained,

$$v_C(t) = v_C(t)|_{forced} + v_C(t)|_{natural}$$
$$= -8e^{-3t} + K_1e^{-4t} + K_2e^{-2t} \quad (16)$$

from where

$$i_L(t) = C\frac{dv_C(t)}{dt} = \frac{1}{4}\left[24e^{-3t} - 4K_1e^{-4t} - 2K_2e^{-2t}\right]$$
$$= \frac{1}{2}\left[\,12e^{-3t} - 2K_1e^{-4t} - K_2e^{-2t}\right] \quad (17)$$

Both initial conditions calculated at step DE.1 and given in (10) and (11) are to be used to determine the unknown constants K_1 and K_2.

$$v_C(t)|_{t=0^+} = -8 + K_1 + K_2 = v_C(t=0^-) = 0 \text{ V}$$

$$i_L(t)|_{t=0^+} = i_C(t)|_{t=0^+} = C\frac{dv_C(t)}{dt}\bigg|_{t=0^+}$$
$$= \frac{1}{2}[\,12 - 2K_1 - K_2] = i_L(t=0^-) = \frac{3}{2} \text{ A}$$

Which results in the equations system,

$$\left.\begin{array}{l} K_1 + K_2 = 8 \\ 2K_1 + K_2 = 9 \end{array}\right\} \Rightarrow \left.\begin{array}{l} K_1 = 1 \\ K_2 = 7 \end{array}\right\}$$

With these values introduced in (16) the final result for the capacitor voltage is,

Complete solution $= v_C(t \geq 0)$

$$= \underbrace{-8e^{-3t}}_{\text{forced}} + \underbrace{e^{-4t} + 7e^{-2t}}_{\text{natural}} \quad (18)$$

STEP DE.6—Output response. As the magnitude defined as output variable is not the voltage drop across the capacitor obtained in (18) but across the resistance of 3Ω, further development is still necessary. Taking into account that $i_L(t>0) = i_C(t>0)$ is the same current that flows across this resistance and Eq. (17), the circuit output response $v_o(t>0)$, still separated into natural and forced components, is

$$v_o(t>0) = Ri_L(t>0) = \frac{3}{2}\left[12e^{-3t} - 2e^{-4t} - 7e^{-2t}\right] = \underbrace{\left[18e^{-3t}\right]}_{\text{forced}} + \underbrace{\left[-3e^{-4t} - \frac{21}{2}e^{-2t}\right]}_{\text{natural}} \quad (19)$$

Once this example analysis is accomplished, the main disadvantages of the method can be highlighted immediately:

a) *Too many variables involved in the analysis.* Whichever variable is defined as the circuit output, the differential equation should be defined in terms of an electrical magnitude that follows the continuity condition. Besides, in order to determine the values of the K_j constants in the natural response to match all the initial conditions, mathematical expressions for all such magnitudes (as many as the circuit order) are required. See (17) in step DE.5.
b) *Limited suitability for higher-order circuits.* Unavoidably, the higher the order the greater the number of initial conditions, unknown constants and natural frequencies to be determined. But, apart from the just mentioned increase in the involved variables whose expression must be derived, the method offers some additional drawbacks: it becomes harder to obtain the corresponding differential equation and more laborious to extract the particular solution, since many derivatives of the function proposed as a solution must be acted on.
c) *Limited versatility regarding to the excitation function.* Should the same circuit topology, with the same components, the same initial conditions but a different source of excitation $v_i(t)$ be of interest, the performed analysis is scarcely useful. Computations have to be redone from step DE.4, because all the following steps, results and values depend on this excitation function, even the natural response.
d) *Non-supported derivation of the characteristic equation.* From the homogeneous differential equation, the student is normally taught to proceed by replacing the n-th time derivative operator by s^n, s being a complex variable, in order to obtain the system characteristic equation. It almost looks like a rule of thumb, since there is no better reason for it than the fact that a solution in the form $K\,e^{st}$ seems to work in first-order circuits and the self-similarity exhibited by an exponential function and its successive derivatives. And the worst of it is that, to have such weak reasons available when facing second-order circuits, text book authors must solve first-order circuits in a different way, so that neither characteristic polynomial and equation nor the complex s variable can be introduced at that moment. In the students' mind, this inconsistent treatment raises doubts about the generality of the method regarding to circuit order.
e) *Too many exceptional cases.* Apart from the different treatment devoted to first-order circuits, at least two other special cases that drift away from the usual procedure must be taught to the students.

The first one appears when the characteristic polynomial real roots are equal; in our example circuit, when $s_1 = s_2$. Then it is stated that an expression for the natural component in the form $[K_1 + tK_2]e^{s_1 t}$ instead of that used in (14) must be attempted.

The second exception arises when the external source frequency ($s_o \epsilon R$) equals a natural frequency in the circuit, i.e. when $v_i(t) = Ae^{s_o t}$ with either $s_o = s_1$ or $s_o = s_2$. Then an alternative expression for the forced response in the form $tKe^{s_o t}$ instead of $Ke^{s_o t}$ has to be proposed and substituted in the non-homogeneous differential equation to determine the value for parameter K and derive the particular solution. Both special cases can take place at the same time, whether $s_o = s_1 = s_2$. If such happens, $[K_1 + tK_2]e^{s_o t}$ is the form for the natural response and $t^2Ke^{s_o t}$ that to be tested as forced response.
f) *Too hard alternative decomposition.* It is sometimes of great interest to separate the complete response into zero-input and zero-state terms. If such is the case, a lot of extra mathematical effort is required indeed.

To calculate the zero-input contribution, the particular solution must be assumed nought and step DE.5 (and following ones) repeated under these conditions, neglecting the forced component in (16) and (17), to get new values for the natural component undetermined constants,

$$\left.\begin{aligned} v_C(t)|_{zi} &= v_C(t)|_{natural} = K_1'\,e^{-4t} + K_2'\,e^{-2t} \\ i_L(t)|_{zi} &= \tfrac{1}{4}\left[-4K_1'\,e^{-4t} - 2K_2'\,e^{-2t}\right] \end{aligned}\right\}$$

Remembering the initial conditions,

$$\left.\begin{aligned} v_C(t=0^+)|_{zi} &= K_1' + K_2' \\ &= v_C(t=0^-) = 0\ \text{V} \\ i_L(t=0^+)|_{zi} &= \tfrac{1}{2}\left[-2K_1' - K_2'\right] \\ &= i_L(t=0^-) = \tfrac{3}{2}\ \text{A} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} K_1' &= -3 \\ K_2' &= 3 \end{aligned}\right\}$$

In this way the zero-input contribution for the output is,

$$\begin{aligned} v_o(t>0)|_{zi} &= 3i_L(t>0) \\ &= \frac{3}{2}\left[-2K_1' e^{-4t} - K_2' e^{-2t}\right] \\ &= 9e^{-4t} - \frac{9}{2}e^{-2t} \end{aligned}$$

To obtain the zero-state contribution, it is needed again to go back to step DE.5 and, although (16) and (17) are still valid, zero initial conditions for both reactive components must be imposed.[5] Once again, new values for the constants have to be determined,

$$\left.\begin{aligned} v_C(t=0^+)|_{zs} &= -8 + K_1'' + K_2'' \\ &= v_C(t=0^-) = 0 \text{ V} \\ i_L(t=0^+)|_{zs} &= \tfrac{1}{2}\left[12 - 2K_1'' - K_2''\right] \\ &= i_L(t=0^-) = 0 \text{ A} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} K_1'' &= 4 \\ K_2'' &= 4 \end{aligned}\right\}$$

So that the zero-state contribution is,

$$\begin{aligned} v_o(t>0)|_{zs} &= \frac{3}{2}\left[12e^{-3t} - 2K_1'' e^{-4t} - K_2'' e^{-2t}\right] \\ &= 18e^{-3t} - 12e^{-4t} - 6e^{-2t} \end{aligned}$$

APPLICATION OF THE LT FOR TIME ANALYSIS. BENEFITS

To solve the identical circuit problem to that in Fig. 2 above applying the Laplace transform method, steps from LT.1 to LT.4 are progressed as follows:

STEP LT.1—Previously computed at step DE.1: $v_C(t=0^-) = 0$ V; $i_L(t=0^-) = 3/2$ A

STEP LT.2—For the transformed circuit in Fig. 5 an internal voltage source connected in series with the inductor is added to incorporate its non-zero initial condition, as stipulated by the rule derived from the Laplace transform action over the derivative voltage-current characteristic in an inductor,

$$v_L(t) = L\frac{di_L(t)}{dt} \xrightarrow{LT} V_L(s) = sL \cdot I_L(s) - Li_L(0^-)$$

STEP LT.3—The analysis of the circuit in the complex-frequency domain should provide the transformed expression for the output magnitude

$$V_o(s) = LT\left[v_o(t)\right],$$

Series connection of voltage sources:

$$V_T(s) = Li_L(0^-) + V_i(s)$$

Series connection of impedances:

$$Z_T(s) = R + sL + \frac{1}{sC} = \frac{s^2LC + sRC + 1}{sC}$$

Total current:

$$\begin{aligned} I_o(s) = \frac{V_T(s)}{Z_T(s)} &= \frac{sC}{s^2LC + sRC + 1} \cdot Li_L(0^-) \\ &+ \frac{sC}{s^2LC + sRC + 1} \cdot V_i(s) \end{aligned}$$

Output function is extracted and impedance values indicated in Fig. 5 are introduced to finally yield,

$$\begin{aligned} V_o(s) = RI_o(s) &= \frac{sRC}{s^2LC + sRC + 1} \cdot Li_L(0^-) \\ &+ \frac{sRC}{s^2LC + sRC + 1} \cdot V_i(s) = \\ &= \underbrace{\frac{6s}{s^2 + 6s + 8} \cdot \frac{3}{4}}_{\text{zero-input}} + \underbrace{\frac{6s}{s^2 + 6s + 8} \cdot V_i(s)}_{\text{zero-state}} \end{aligned} \tag{20}$$

STEP LT.4—From now on, all the analytical expressions in time domain are assumed for $t > 0$. As previously explained, this method provides the complete response naturally split into zero-input and zero-state components,

$$v_o(t) = v_o(t)|_{zi} + v_o(t)|_{zs}$$

with the advantage that the zero-input response computation is completely independent of the actual excitation source,

$$\begin{aligned} v_o(t)|_{zi} &= LT^{-1}\left[\frac{9s}{2} \cdot \frac{1}{s^2 + 6s + 8}\right] \\ &= LT^{-1}\left[\frac{9}{s+4} + \frac{-9/2}{s+2}\right] \\ &= 9\left[e^{-4t} - \frac{1}{2}e^{-2t}\right] u(t) \end{aligned} \tag{21}$$

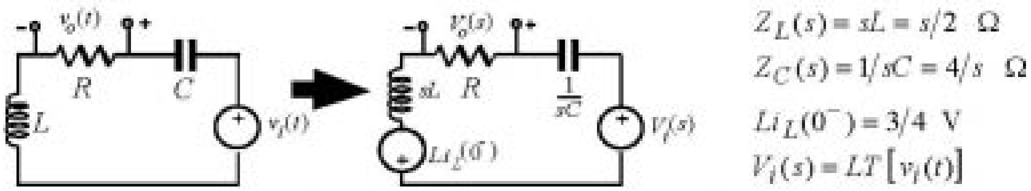


Fig. 5. Circuit transformation into the complex-frequency domain.

For the zero-state component the function of excitation must be considered,

$$v_i(t) = e^{-3t}u(t) \xrightarrow{LT} V_i(s) = \frac{1}{s+3}$$

thus yielding,

$$\begin{aligned} v_o(t)|_{zs} &= LT^{-1}\left[\frac{1}{s+3}\cdot\frac{6s}{s^2+6s+8}\right] \\ &= LT^{-1}\left[\frac{18}{s+3}-\frac{12}{s+4}-\frac{6}{s+2}\right] = \\ &= 6\left[3e^{-3t}-2e^{-4t}-e^{-2t}\right]u(t) \end{aligned} \tag{22}$$

And of course, the addition of both components given in (21) and (22) provides the complete response,

$$\begin{aligned} v_o(t) &= v_o(t)|_{zi} + v_o(t)|_{zs} \\ &= \left[18e^{-3t}-3e^{-4t}-\frac{21}{2}e^{-2t}\right]u(t) \end{aligned} \tag{23}$$

Anybody can deny that, (19) and (23) being the same result, it has been much faster to derive (23) using the proposed method, and even simpler from a mathematical point of view, since it is always easier to solve linear equations than differential equations.

These benefits are not due to the specific circuit chosen as an example for comparison purposes. It is the whole procedure that presents many advantages over the method classically taught and implemented.

a) *Output magnitude expression is directly calculated.* Once the circuit has been transformed into the s-domain, its analysis can be aimed at obtaining the transformed expression $Y(s)$ of the output magnitude, wherever it is defined. Neither time nor effort are wasted in computing expressions for the remaining variables under continuity restraints, since the initial conditions are included as internal sources from the very beginning, when the circuit is transformed.

b) *Method generality*. The proposed method is suitable for any source of excitation, provided that its Laplace transform is known or can be calculated. Besides, as this information only affects one of the last steps in the analysis, it can be changed quite easily without requiring too much recalculation: only the zero-state contribution would need to be modified.

   It is also appropriate for any circuit order: there is no need to apply a different approach for first-order circuits, and application for higher-order is not at the cost of much more work.

   Moreover, all the exceptional cases reported as drawbacks in the preceding section are perfectly encompassed in the described methodology at the sight of the Laplace transform pair

$$\begin{aligned} f(t) &= \frac{t^{n-1}\cdot e^{-at}}{(n-1)!}u(t) \xrightarrow{LT} F(s) \\ &= \frac{1}{(s+a)^n}\,; \quad n = 1,2,3... \end{aligned} \tag{24}$$

   Indeed, all of them are cases where the denominator polynomial $d(s)$ in (1) exhibits poles with multiplicity greater than one (either coming from the characteristic polynomial or from the excitation source transform), so that partial-fractions with the form of the right term in (24) arise when computing the inverse transform.

c) *Concept understanding*. From the Laplace transform perspective, there is no problem for the students coming to terms with the complex s-variable.

d) *Easy alternative decomposition.* When it is desired to separate the complete response into natural and forced components, the task is quite easy and most of the work has been done beforehand.

   Rewriting (20) in the form

$$\begin{aligned} V_o(s) &= \frac{6s}{s^2+6s+8}\cdot\left[V_i(s)+\frac{3}{4}\right] \\ &= \frac{6s}{(s+2)(s+4)}\cdot\left[\frac{1}{s+3}+\frac{3}{4}\right] \\ &= \frac{A}{s+2}+\frac{B}{s+4}+\frac{C}{s+3} \end{aligned}$$

   it becomes clear that $s=-2,-4$ are the system poles (the roots of the characteristic polynomial) and the inverse transform of the fractions containing these poles will produce the natural response. On its side, $s=-3$ is a pole of $Y(s)$ introduced by the excitation source $X(s) = LT[x(t)]$. The inverse transform of partial-fractions containing such extra poles coming from $X(s)$ will produce the forced response.

   Therefore, both terms can be distinguished in the final result (23) with no need for a new partial-fraction expansion and inverse transform.[6]

$$\begin{aligned} v_o(t) &= \left[18e^{-3t}-3e^{-4t}-\frac{21}{2}e^{-2t}\right]u(t) \\ &= \underbrace{\left[18e^{-3t}\right]u(t)}_{\text{forced}} + \underbrace{\left[-3e^{-4t}-\frac{21}{2}e^{-2t}\right]u(t)}_{\text{natural}} \end{aligned}$$

THE USUAL APPROACH FOR AC STEADY-STATE ANALYSIS. DRAWBACKS

Sinusoidal steady-state analysis is a task where there are many few degrees of freedom. As only the forced response is pursued, initial conditions,

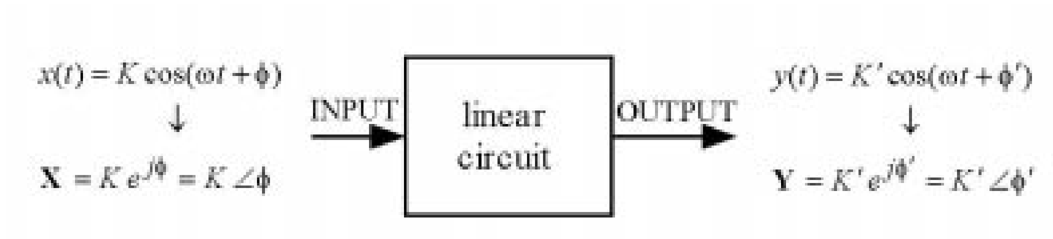


Fig. 6. Conceptual scheme of the phasor method basis.

natural responses, zero-input responses and transients can be completely neglected. Moreover, since the excitation is forced to be a sinusoidal function and the circuits are assumed to be linear, all the electrical waveforms involved in the analysis will also be sinusoidal functions, all of them with the same frequency: the excitation frequency.

From this perspective, ac steady-state analysis should be a really simple subject to cope with.

However, once again, the usual approach for teaching/learning how to analyse circuits under ac steady-state conditions (the so-called phasor method) introduces new techniques, new concepts and new mathematical tools that could be avoided to students. It does not even make use of previously acquired skills: for a sinusoidal input signal $x(t) = K\cos(\omega t + \phi)$ a forced response as $y(t)|_{forced} = K_A \cos\omega t + K_B \sin\omega t$ with its successive time derivatives could be introduced in the differential equation (step DE.4) to determine constants K_A and K_B. From here, the output steady-state waveform could be extracted.

Figure 6 illustrates the underlying idea that is the basis of the phasor method. Independent of how many sinusoidal sources (assumed of the same frequency) the circuit contains and which is the magnitude defined as output, all the electrical signals in the circuit will share the same frequency but exhibit different values for amplitude and phase. These parameters subjected to change, amplitude and phase, are used to construct the corresponding phasors.

To illustrate the phasor method, the same second-order network in Fig. 4 above, with the same component values, but now considering sinusoidal excitation with amplitude K, frequency $\omega = 1$ rad/s and zero phase,

$$v_i(t) = K\cos(\omega t + \phi) = K\cos t \text{ V} \tag{25}$$

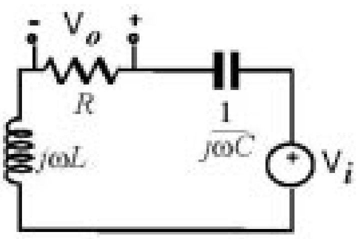
is considered. Assuming steady-state reached, the circuit is analysed, step by step, with the classical approach.

Through this circuit example they are shown both the way to derive a certain magnitude defined as output variable and the method to obtain the circuit frequency response function. Method extension to circuits of higher-order and/or containing more sinusoidal excitation sources (always with the same frequency) is direct.

STEP PM.1—Equivalent phasor representation. To analyse the sinusoidal steady-state using the phasor method, signals and components must be first transformed. The former are represented by the corresponding phasor, and the later are replaced by their respective complex impedances, calculated depending on their passive values and the sinusoidal signal frequency.

STEP PM.2—Circuit analysis with the phasor method. In this simple example the analysis involves a series connection of impedances plus a voltage divider,

$$\begin{aligned} \mathrm{Z}_t(j\omega) &= \mathrm{Z}_C(j\omega) + \mathrm{Z}_R(j\omega) + \mathrm{Z}_L(j\omega) \\ &= \frac{1}{j\omega C} + R + j\omega L \\ \mathrm{V}_o(j\omega) &= \mathrm{V}_i(j\omega)\cdot\frac{\mathrm{Z}_R(j\omega)}{\mathrm{Z}_t(j\omega)} \\ &= \mathrm{V}_i(j\omega)\cdot\frac{R}{\dfrac{1}{j\omega C} + R + j\omega L} \\ &= \frac{\mathrm{V}_i(j\omega)\cdot R}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)} \end{aligned} \tag{28}$$



$$v_i(t) \to \mathrm{V}_i = K\angle\phi = K e^{j\phi} = K\angle 0^\circ \tag{26}$$

$$\left.\begin{aligned} R &\to \mathrm{Z}_R(j\omega) = R = 3\ \Omega \\ L &\to \mathrm{Z}_L(j\omega) = j\omega L = \frac{j}{2}\ \Omega \\ C &\to \mathrm{Z}_C(j\omega) = 1/j\omega C = -4j\ \Omega \end{aligned}\right\} \tag{27}$$

Fig. 7. Equivalent phasor representation for ac steady-state.

STEP PM.3— Output variable steady-state derivation. In the output phasor $V_o(j\omega)$ expression (28) the input phasor derived in (26) and the values for the components and excitation frequency have to be introduced, thus yielding,

$$V_o = K' \angle \phi' = \frac{3K\angle 0^\circ}{3 + j\left(\frac{1}{2} - 4\right)} = \frac{3K}{3 - \frac{7}{2}j}$$
$$= \frac{6K}{6 - 7j} = \frac{6K}{\sqrt{85}} \angle 49,4^\circ$$

what has still to be converted back to the time domain, recovering a sinusoidal function with the corresponding frequency $\omega = 1$ rad/s. Taking into account the trigonometric relation

$$\cos(a \pm b) = \cos a \cdot \cos b \mp \sin a \cdot \sin b$$

the steady-state for the output variable is finally

$$v_o(t) = K' \cos(\omega t + \phi') = \frac{6K}{\sqrt{85}} \cos(t + 49,4^\circ)$$
$$= \frac{K}{85}[36 \cos t - 42 \sin t]\mathrm{V} \qquad (29)$$

STEP PM.4—Frequency response. Going back to the output phasor $V_o(j\omega)$ expression (28) the corresponding network function (a voltage transfer function in the example), defined as the ratio of the output phasor to the input phasor, both in terms of the real frequency variable ω, can be obtained

$$\mathrm{H}(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{R}{R + j\left(\omega L - \frac{1}{\omega C}\right)}$$
$$= \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC} = \frac{j6\omega}{(8 - \omega^2) + j6\omega} \qquad (30)$$

From where magnitude and phase responses of the network can be extracted,
Magnitude response:

$$|H(j\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$
$$= \frac{6\omega}{\sqrt{(8 - \omega^2)^2 + (6\omega)^2}} \qquad (31)$$

Phase response:

$$\angle H(j\omega) = 90^\circ - \arctan\left[\frac{\omega RC}{1 - \omega^2 LC}\right]$$
$$= 90^\circ - \arctan\left[\frac{6\omega}{8 - \omega^2}\right] \qquad (32)$$

As clearly shown in step PM.3 the phasor method only provides the particular solution of the zero-state response. Taking this into account the main disadvantages of the method are all related to the very restricted conditions of applicability.

a) *Restriction on the frequency*. Should the circuit contain several excitation sources, in order to perform the phasor analysis just once, all the sources must share the same frequency value. Otherwise superposition principle has to be applied in the time domain and several transformations and analyses (as many as different frequencies), from step PM.1 to step PM.3, need to be carried out.
b) *Restriction on excitation function*. A sinusoidal function is imposed as circuit excitation by the phasor method. Note that such a condition is in the analysis from the very beginning to derive the input phasor and the circuit impedances (step PM.1). Should the same circuit be excited by a source departing from the sinusoidal function, the analysis is completely useless.
c) *Restriction to steady-state*. As the method only provides the forced response, it does not allow any information to be gathered related to natural frequencies, transient response and/or damping pattern.
d) *No information about stability*. As the method does not extract the circuit natural frequencies, poles location and stability are never taken into account. It does not seem to make much sense to waste time and effort in analysing an unstable system.

APPLICATION OF THE LT FOR AC STEADY-STATE ANALYSIS. BENEFITS

The result (20) previously obtained is now recovered to perform the same ac steady-state analysis, but now from the LT perspective. Remembering what was explained above in step LT.5, there is no need either to recognize the part-fraction related to steady-state or to compute the inverse LT transform.

STEP LT.5—Assuming steady-state under sinusoidal excitation in (20) and hence neglecting the zero-input contribution yields the transfer function.

$$V_o(s)|_{zs} = \frac{sRC}{s^2 LC + sRC + 1} \cdot$$
$$V_i(s) = \frac{6s}{s^2 + 6s + 8} \cdot V_i(s)$$
$$\Rightarrow \quad H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sRC}{s^2 LC + sRC + 1}$$
$$= \frac{6s}{s^2 + 6s + 8} \qquad (33)$$

Under replacement $s = j\omega = j$ (remember that

$\omega = 1$ rad/s in our example), magnitude and phase responses are extracted,

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{j6\omega}{8-\omega^2+j6\omega} = \frac{6j}{7+6j}$$
$$\Rightarrow \begin{cases} |H(j\omega)| = 6/\sqrt{85} \\ \angle H(j\omega) = 90^\circ - \arctan(6/7) = 49,4^\circ \end{cases} \tag{34 and 35}$$

Steady-state output response is straightforwardly derived following instructions in Fig. 1,

$$\begin{aligned} v_o(t)|_{ss} &= K' \cos(\omega t + \phi') \\ &= K|H(j\omega)| \cos[\omega t + \phi + \angle H(j)] \\ &= \frac{6K}{\sqrt{85}} \cos(t + 49,4^\circ)\text{V} \end{aligned} \tag{36}$$

Once again, results in (29) and (36) are exactly equal, but obtaining (36) has been faster and has not required any extra knowledge.

Some other benefits arising from this approach stand out:

a) *Analysis profitability*. Should it be of interest, natural response and transient performance could be derived from the same analysis (recovering terms that have been neglected on the way), as well as the output corresponding to any other excitation function. None of such items can be achieved with the phasor approach.
b) *Global understanding*. Deriving the frequency response from the transfer function under replacement $s = j\omega$ allows an easy understanding. From this approach perspective, ac steady-state is viewed as a special case where the s complex-frequency projects on the imaginary axis $j\omega$, which provides a much better insight into Bode plots, their meaning, utility and connection to transfer functions.
c) *Unnecessary recurrences*. According to the previous paragraph, steady-state impedances $Z(j\omega)$ can be derived from Laplace transformed impedances $Z(s)$ as a particular case, which means that a lot of repetitive work can be avoided by students (and teachers). Results such as series and parallel connection of impedances, nodal and loop analysis techniques, network theorems, Thévenin and Norton equivalents, etc. do not require to be explained so many times with different notation (now $j\omega$ and then s), but just once (in s).
d) *Universal tool*. Students could be taught to use a single mathematical tool to analyse in any domain (time or frequency) any circuit of any order with any excitation, thereby avoiding the nightmare of learning so many different methods with so limited applicability range.
e) *Time-frequency performance connection*. Looking back to expression (1) and checking the common denominator in both zero-state and zero-input, reveals a new outstanding conceptual advantage: there is an intrinsic connection between the damping pattern in the natural response (transient behaviour) and the frequency response characteristics (filtering performance), since both are direct consequences of the circuit characteristic polynomial order and roots. Such conceptual connection remains irretrievably hidden when the differential equations method is used to obtain the former and phasor method to derive the latter.

DISCUSSION AND CONCLUSIONS

A simple second-order circuit has illustrated how to use the Laplace transform as a unique universal tool when teaching basic circuit analysis. This approach allows the extraction of complete information about time domain behaviour and frequency domain performance, but with the advantage of avoiding unnecessary mathematical skills such as solving differential equations and working with phasors.

Some other conceptual advantages and teaching/learning benefits have also been demonstrated along the way.

END NOTES

[1] A single external source $x(t)$ has been considered in (1) for the sake of clarity, but the linearity of the circuit ensures that, for the case of multiple external sources, the zero-state contribution can be expressed as

$$Y_{zs}(s) = \sum_m \left[\frac{N_m(s)}{D_m(s)} \cdot X_m(s)\right]$$

and will still be easily recognisable thanks to the $X_m(s)$ factors.

[2] However, both contributions can be joined and inversely transformed as a whole (thus demanding less mathematical effort in obtaining the partial-fraction expansion) if such breaking down is not of interest. It will be shown in section 4 that, in any case, it is quite easy from (2) to obtain the alternative splitting into natural and forced components.

[3] Such conjugate symmetry can be demonstrated, if needed, from the Laplace Transform definition,

$$F(s) = LT[f(t)] = \int_0^\infty f(t)e^{-st}dt; s \in X \Rightarrow [F(s)]^* = F(s*)$$

provided that $f(t)$ is a real function

[4] From a didactic point of view, the best way to proceed in such a case would be to apply superposition principle, i.e. as many partial analyses as different sources the circuit contains. Each partial analysis would lead to an input definition (depending on the source being considered) and therefore a transfer function definition $H_m(s)$, so that the corresponding partial contribution to the steady-state response is calculated as in (9). Adding at the end all these partial contributions, the complete steady-state response $y_{ss}(t)$ is derived,

$$y_{ss}(t) = \sum_m K'_m \cos(\omega t + \phi'_m)$$
$$= \sum_m K_m \left|H_m(j\omega)\right| \cos[\omega t + \phi_m + \angle H_m(j\omega)]$$

[5] To tell the whole truth, if the previous result (19) for the complete response had been previously calculated, only one of these contributions might be compulsory obtained. The other one could be derived as a difference, since in any case the sum of zero-input and zero-state produces the complete response. Nevertheless, even if one of these calculations can be avoided, the task required to obtain the remaining one is still quite tedious.

[6] Even if the circuit had poles with multiplicity $m > 1$, the forced response would still be recognisable as the term arising from the inverse transform of that partial-fraction in the form of the right term in Eq.(24) with maximum exponent, i.e. $n = m$.

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